Characteristic polynomial, eigenvalues, and eigenvectors 2

For each of the following matrices, the following is requested:

1. Find the characteristic polynomial.
2. Find the eigenvalues and the associated eigenvectors.

c)

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 4 \\ 0 & 0 & 7 \end{pmatrix}$$

d)

$$\begin{pmatrix} -2 & -1 & -1 \\ 4 & -8 & -6 \\ -4 & 11 & 9 \end{pmatrix}$$

e)

$$\begin{pmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix}$$

Solution

c) Characteristic polynomial

The characteristic polynomial of A is obtained by computing

$$p(\lambda) = \det(A - \lambda I)$$

In this case,

$$A - \lambda I = \begin{pmatrix} 1-\lambda & 2 & 3 \\ 0 & 3-\lambda & 4 \\ 0 & 0 & 7-\lambda \end{pmatrix}$$

Since the matrix is triangular, its determinant is the product of the diagonal elements:

$$p(\lambda) = (1-\lambda)(3-\lambda)(7-\lambda)$$

Thus, the characteristic polynomial of A is:

$$p(\lambda) = (1-\lambda)(3-\lambda)(7-\lambda)$$

Eigenvalues and eigenvectors

The eigenvalues are the values of λ that nullify the characteristic polynomial:

$$\lambda_1 = 1, \quad \lambda_2 = 3, \quad \lambda_3 = 7$$

Eigenvalue $\lambda_1 = 1$

To find the associated eigenvector, we solve

$$(A - I)\mathbf{v} = 0$$

In matrix form:

$$A - I = \begin{pmatrix} 0 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 6 \end{pmatrix}$$

The resulting linear system is:

$$\begin{cases} 2y + 3z = 0 \\ 2y + 4z = 0 \\ 6z = 0 \end{cases}$$

From the last equation, $z = 0$. Substituting into the other equations, $y = 0$, and x can take any value. Taking $x = 1$, an associated eigenvector is:

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Eigenvalue $\lambda_2 = 3$

For this eigenvalue, we solve:

$$(A - 3I)\mathbf{v} = 0$$

In matrix form:

$$A - 3I = \begin{pmatrix} -2 & 2 & 3 \\ 0 & 0 & 4 \\ 0 & 0 & 4 \end{pmatrix}$$

The resulting linear system is:

$$\begin{cases} -2x + 2y + 3z = 0 \\ 4z = 0 \\ 4z = 0 \end{cases}$$

From the last equation, $z = 0$. The first equation reduces to $-2x + 2y = 0$, i.e., $x = y$. Taking $x = 1$, we obtain $y = 1$, with $z = 0$. An associated eigenvector is:

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Eigenvalue $\lambda_3 = 7$

For this eigenvalue, we solve:

$$(A - 7I)\mathbf{v} = 0$$

In matrix form:

$$A - 7I = \begin{pmatrix} -6 & 2 & 3 \\ 0 & -4 & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

The resulting linear system is:

$$\begin{cases} -6x + 2y + 3z = 0 \\ -4y + 4z = 0 \\ 0 = 0 \end{cases}$$

From the second equation, $-4y+4z = 0$ implies $y = z$. The first equation rewrites as $-6x+2y+3z = 0$, which with $y = z$ gives $-6x + 5z = 0$, i.e., $x = \frac{5}{6}z$. Taking $z = 6$, we obtain $x = 5$ and $y = 6$. An associated eigenvector is:

$$\mathbf{v}_3 = \begin{pmatrix} 5 \\ 6 \\ 6 \end{pmatrix}$$

d) Characteristic polynomial

The characteristic polynomial of A is obtained by computing

$$p(\lambda) = \det(A - \lambda I)$$

For the matrix

$$A - \lambda I = \begin{pmatrix} -2-\lambda & -1 & -1 \\ 4 & -8-\lambda & -6 \\ -4 & 11 & 9-\lambda \end{pmatrix}$$

we have:

$$\begin{aligned} p(\lambda) &= (-2-\lambda)\det\begin{pmatrix} -8-\lambda & -6 \\ 11 & 9-\lambda \end{pmatrix} - (-1)\det\begin{pmatrix} 4 & -6 \\ -4 & 9-\lambda \end{pmatrix} - \det\begin{pmatrix} 4 & -8-\lambda \\ -4 & 11 \end{pmatrix} \\ &= (-2-\lambda)\Big[(-8-\lambda)(9-\lambda)+66\Big] + \Big[4(9-\lambda)-24\Big] \\ &\quad - \Big[44-4(8+\lambda)\Big] \\ &= (-2-\lambda)\Big[\lambda^2-\lambda-6\Big] \\ &= -(\lambda+2)(\lambda^2-\lambda-6) \\ &= -(\lambda+2)(\lambda-3)(\lambda+2) \\ &= -(\lambda+2)^2(\lambda-3) \end{aligned}$$

Therefore, the characteristic polynomial of A is:

$$p(\lambda) = -(\lambda+2)^2(\lambda-3)$$

Eigenvalues and eigenvectors

The eigenvalues are the values of λ that nullify the characteristic polynomial:

$$\lambda_1 = -2 \quad \text{(multiplicity 2)}, \quad \lambda_2 = 3$$

Eigenvalue $\lambda_1 = -2$

To find an eigenvector associated with $\lambda = -2$, we solve

$$(A-(-2)I)\mathbf{v} = (A+2I)\mathbf{v} = 0$$

where

$$A+2I = \begin{pmatrix} 0 & -1 & -1 \\ 4 & -6 & -6 \\ -4 & 11 & 11 \end{pmatrix}$$

The resulting system is:

$$\begin{cases} -y-z=0, \\ 4x-6y-6z=0, \\ -4x+11y+11z=0. \end{cases}$$

From the first equation, we have $y = -z$. Substituting into the second, we get $4x = 0$, which implies $x = 0$. The third equation is automatically satisfied. Taking $z = 1$, an associated eigenvector is:

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

Eigenvalue $\lambda_2 = 3$

For this eigenvalue, we solve:

$$(A-3I)\mathbf{v} = 0$$

where

$$A-3I = \begin{pmatrix} -5 & -1 & -1 \\ 4 & -11 & -6 \\ -4 & 11 & 6 \end{pmatrix}$$

The system of equations is:

$$\begin{cases} -5x - y - z = 0, \\ 4x - 11y - 6z = 0, \\ -4x + 11y + 6z = 0. \end{cases}$$

From the first equation, we deduce:

$$5x + y + z = 0 \quad \Rightarrow \quad y = -5x - z.$$

Substituting into the second:

$$4x - 11(-5x - z) - 6z = 4x + 55x + 11z - 6z = 59x + 5z = 0,$$

which implies:

$$z = -\frac{59}{5}x.$$

Then,

$$y = -5x - \left(-\frac{59}{5}x\right) = \frac{34}{5}x.$$

Taking $x = 5$, we get:

$$\mathbf{v}_2 = \begin{pmatrix} 5 \\ 34 \\ -59 \end{pmatrix}$$

e) Characteristic polynomial

The characteristic polynomial of A is obtained by computing

$$p(\lambda) = \det(A - \lambda I)$$

For the matrix

$$A - \lambda I = \begin{pmatrix} -\lambda & -1 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{pmatrix}$$

we have:

$$\begin{aligned} p(\lambda) &= \det \begin{pmatrix} -\lambda & -1 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{pmatrix} \\ &= -\lambda \det \begin{pmatrix} 2-\lambda & 1 \\ 2 & 3-\lambda \end{pmatrix} + \det \begin{pmatrix} 1 & 1 \\ 2 & 3-\lambda \end{pmatrix} - \det \begin{pmatrix} 1 & 2-\lambda \\ 2 & 2 \end{pmatrix} \\ &= -\lambda\Big[(2-\lambda)(3-\lambda) - 2\cdot 1\Big] + \Big[(3-\lambda) - 2\Big] - \Big[2 - 2(2-\lambda)\Big] \\ &= -\lambda\Big[\lambda^2 - 5\lambda + 4\Big] + (1-\lambda) - \Big[2\lambda - 2\Big] \\ &= -\lambda(\lambda-1)(\lambda-4) - (\lambda-1) - 2(\lambda-1) \\ &= -\lambda(\lambda-1)(\lambda-4) - 3(\lambda-1) \\ &= -(\lambda-1)\Big[\lambda(\lambda-4) + 3\Big] \\ &= -(\lambda-1)\Big[\lambda^2 - 4\lambda + 3\Big] \\ &= -(\lambda-1)^2(\lambda-3) \end{aligned}$$

Therefore, the characteristic polynomial of A is:

$$p(\lambda) = -(\lambda-1)^2(\lambda-3)$$

Eigenvalues and eigenvectors

The eigenvalues are the values of λ that nullify the characteristic polynomial:

$$\lambda_1 = 1 \quad \text{(multiplicity 2)}, \quad \lambda_2 = 3$$

Eigenvalue $\lambda_1 = 1$

To find the associated eigenvectors, we solve

$$(A - I)\mathbf{v} = 0$$

where

$$A - I = \begin{pmatrix} -1 & -1 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix}.$$

The only resulting condition is:

$$x + y + z = 0.$$

This defines a two-dimensional eigenspace, so we can choose two linearly independent eigenvectors. For example, a basis for this subspace is:

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

Eigenvalue $\lambda_2 = 3$

For this eigenvalue, we solve:

$$(A - 3I)\mathbf{v} = 0$$

where

$$A - 3I = \begin{pmatrix} -3 & -1 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{pmatrix}.$$

The system of equations is:

$$\begin{cases} -3x - y - z = 0, \\ x - y + z = 0, \\ 2x + 2y = 0. \end{cases}$$

From the third equation, $2x + 2y = 0$ implies $y = -x$. Substituting into the second:

$$x - (-x) + z = 2x + z = 0 \quad \Rightarrow \quad z = -2x.$$

Taking $x = 1$, we obtain:

$$\mathbf{v}_3 = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}.$$